Maxima and Minima

(7.5) **Definition.** Let a function f be defined on $[a, b]$. f is said to have an **absolute maximum** on $[a, b]$ if there is a number c belonging to $[a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. Similarly, f is said to have an **absolute minimum** on $[a, b]$ if there is a number $d \in [a, b]$ such that $f(d) \leq f(x)$ for all $x \in [a, b]$. The numbers $f(c)$ and $f(d)$ are respectively called **absolute maximum** and **absolute minimum values** of f on $[a, b]$. (Some authors use the word **global** instead of **absolute**).

The function f is said to have *****relative maximum** at $c \in]a, b[$ if there exists a number $\delta > 0$ such that $[c - \delta, c + \delta] \subset]a, b[$ and $f(c)$ is the absolute maximum value of f on $[c - \delta, c + \delta] \subset]a, b[$. i.e.,

$$f(c) \geq f(x) \text{ for all } x \in [c - \delta, c + \delta].$$

$f(c)$ is called the **relative maximum value** of f at c.

The function f is said to have a **relative minimum** at $d \in]a, b[$ if there exists a number $\varepsilon > 0$ such that $[d - \varepsilon, d + \varepsilon] \subset]a, b[$ and $f(d)$ is the absolute minimum value of f on $[d - \varepsilon, d + \varepsilon]$; that is

$$f(d) \leq f(x) \quad \text{for all} \quad x \in [d - \varepsilon, d + \varepsilon].$$

$f(d)$ is called the **relative minimum** value of f at d.

The term (relative) **extreme values (extrema)** is used to refer to either a (relative) maximum value or a (relative) minimum value.

A **critical point** for f is any point c in the domain of f at which $f'(c) = 0$ or f is not differentiable at c. The critical points where $f'(x) = 0$ are called **stationary points.**

* Other terminology being used is '**local maximum (minimum)**'.

Concavity

(7.8) Definition. The graph of a function $y = f(x)$ is said to be **concave up** in an interval $]\, a,\ b\, [$ if and only if it lies above every tangent line at the points between $(a, f(a))$ and $(b, f(b))$ on the curve.

Similarly, $y = f(x)$ is **concave down** in an open interval if and only if its graph lies below every tangent line at all points of the open interval.

The function $y = f(x)$ is said to be **concave up (down) at a point** c of its domain if it is concave up (down) in some open interval containing c.

We known that if $f''(x) > 0$ in some interval, then $f'(x)$ is an increasing function. Since $f'(x)$ is slope of the tangent line, and if $f'(x)$ is an increasing function, then as a point P moves from left to right along the curve (Figure 7.5), the slope of the tangent line to the curve increases. Thus the curve is **concave up.**

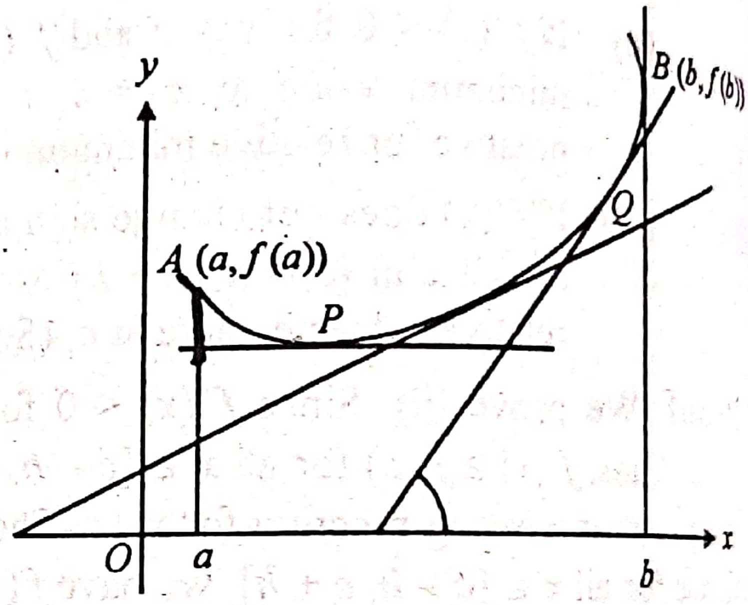

Figure 7.5
Concave up

Similarly, if $f''(x) < 0$ in an interval, then $f'(x)$ is a decreasing function. In this case as a point P moves from left to right along the curve, the slope of the tangent line to the curve decreases (Figure 7.6). The curve is **concave down** under such circumstances.

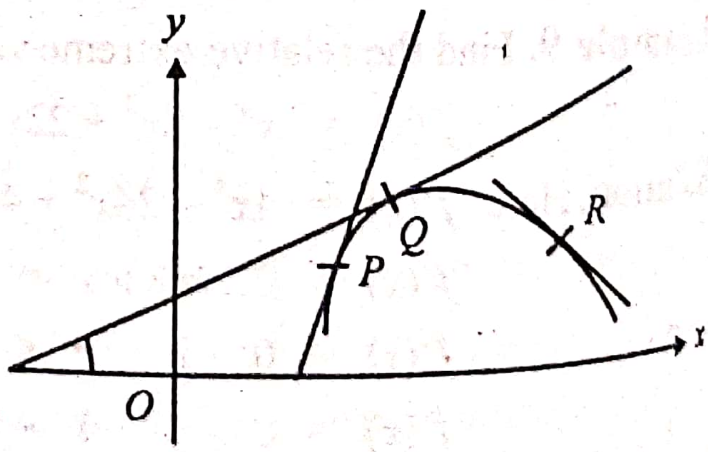

Figure 7.6
Concave down

We summarize these results in

(7.9) Theorem. If $y = f(x)$ possesses continuous second order derivatives in an interval $]a, b[$, then

(i) $y = f(x)$ is concave up in $]a, b[$ if and only if $f''(x) > 0$ for all x belonging to $]a, b[$.

(ii) $y = f(x)$ is concave down in $]a, b[$ if and only if $f''(x) < 0$ for all x belonging to $]a, b[$.

(iii) $y = f(x)$ is concave up at $c \in]a, b[$ if $f''(c) > 0$ and it is concave down at c if $f''(c) < 0$.

Example 10. Find the intervals for which the curve

$$y = x^4 - 6x^3 + 12x^2 + 5x + 7$$

(i) faces up (ii) faces down

Solution. $$\frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$$

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24 = 12(x-1)(x-2)$$

For $x < 1$, $\frac{d^2y}{dx^2} > 0$. When $x = 1$, $\frac{d^2y}{dx^2} = 0$

For $1 < x < 2$, $\frac{d^2y}{dx^2} < 0$. When $x = 2$, $\frac{d^2y}{dx^2} = 0$

For $x > 2$, $\frac{d^2y}{dx^2} > 0$.

Thus the curve faces up in the intervals $x < 1$, $x > 2$ and faces down on $]1, 2[$.

(7.10) Definition. Point of Inflection. A point $x = c$ on a curve $y = f(x)$ is called a **point of inflection** if f is concave up on one side of c and concave down on the other side of c and f is continuous at $x = c$.

The tangent line to a curve at a point of inflection P always crosses the curve because the curve must face up on one side of P and be above the tangent line there, while the curve must face down and be below the tangent line on the other side.

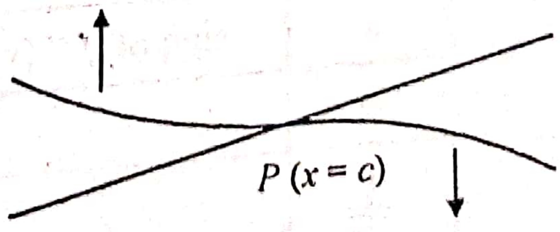

Figure 7.7

The curve $y = f(x)$ changes the direction at a point of inflection P. Hence, if f'' is continuous in an interval including P, then by Theorem 7.9, $f'' \geq 0$ on one side of P and $f'' \leq 0$ on the other side of P. This can only happen if $f'' = 0$ at P. Hence we have the following: